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Advanced Linear Algebra (MA 409)
Problem Sheet - 4

Linear Dependence and Linear Independence

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1. Label the following statements as true or false.
   (a) Any set containing the zero vector is linearly dependent.
   (b) The empty set is linearly dependent.
   (c) Subsets of linearly dependent sets are linearly dependent.
   (d) Subsets of linearly independent sets are linearly independent.
   (e) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

2. Determine whether the following sets are linearly dependent or linearly independent.
   (a) $\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$
   (b) $\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\}$ in $P_3(\mathbb{R})$
   (c) $\{(1, -1, 2), (2, 0, 1), (-1, 2, -1)\}$ in $\mathbb{R}^3$
   (d) $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$
   (e) $\{x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + x^3 + 4x^2 + 8x\}$ in $P_4(\mathbb{R})$

3. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

4. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

5. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

6. Recall that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

7. Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.
   (a) Prove that if $F = \mathbb{R}$, then S is linearly independent.
   (b) Prove that if F has characteristic 2, then S is linearly dependent.

8. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

9. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many vectors are there in span(S)? Justify your answer.

10. Let V be a vector space over a field of characteristic not equal to two.

    (a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

    (b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

    (c) Discuss the part (b) when V is a vector space over the field with two elements. [Hint : If u, v and w are linearly independent, show that $u + v, v + w$ and $w + v$ are linearly independent, provided $1 + 1 \neq 0$. Show by an example that the condition $1 + 1 \neq 0$ cannot be dropped.]

11. Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \ldots, u_n$ in S such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

12. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in span(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \leq k < n)$.

13. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

14. Let M be a square upper triangular matrix with nonzero diagonal entries. Prove that the columns of M are linearly independent.

15. Let S be a set of nonzero polynomials in P(F) such that no two have the same degree. Prove that S is linearly independent.

16. Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent.

17. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

18. Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

19. Let V be the vector space of all 2×2 matrices over the field F. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$$

    (a) Prove that W_1 and W_2 are subspaces of V.

    (b) Find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

    (c) Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j.

20. Show that $\{1, \sqrt{2}\}$ and $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$ are linearly independent over $\mathbb{Q}$ and that $\{\sqrt{2}, \sqrt{3}, \sqrt{12}\}$ is linearly dependent over $\mathbb{Q}$.

21. Let $x_1, x_2, \ldots, x_k$ be vectors in F^n and let y_i be the vector in F^{n-1} formed by the first $n-1$ components of x_i for $i = 1, 2, \ldots, k$. Show that if $y_1, y_2, \ldots, y_k$ are linearly independent in F^{n-1} then $x_1, x_2, \ldots, x_k$ are linearly independent in F^n. Is the converse true? Why?

22. Let A be a linearly independent set and $y \notin A$. Prove that $A \cup \{y\}$ is linearly dependent iff $y \in Sp(A)$. Show by an example that linear independence of A cannot be dropped.

23. Find all the maximal linearly independent subsets of $\{x_1, x_2, \ldots, x_5\}$ where $x_1 = (1, 1, 0, 1)$, $x_2 = (1, 2, -1, 0)$, $x_3 = (1, 0, 1, 2)$,$x_4 = (0, 1, 1, 1)$ and $x_5 = (2, 0, 2, 4)$ in $\mathbb{R}^4$.

24. Let A be a linearly independent subset of a subspace S. If $x \notin S$, show that $A \cup \{x\}$ is linearly independent. If $B \subseteq V - S$ and B is linearly independent, does it follow that $A \cup B$ is linearly independent?

25. Let $Sp(A) = S$. Then show that no proper subset of A generates S iff A is linearly independent.

26. Give geometric characterizations of $\{x_1, x_2\}$ and $\{x_1, x_2, x_3\}$ being linearly independent in $\mathbb{R}^3$.

    [Hint : collinear/coplanar]

27. (a) For what values of α are the vectors $(0, 1, \alpha)$, $(\alpha, 1, 0)$ and $(1, \alpha, 1)$ in $\mathbb{R}^3$ linearly independent?

    (b) Determine all the values of α and β for which the vectors $(\alpha, \beta, \beta, \beta)$, $(\beta, \alpha, \beta, \beta)$, $(\beta, \beta, \alpha, \beta)$ and $(\beta, \beta, \beta, \alpha)$ of $\mathbb{R}^4$ are linearly dependent.

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